

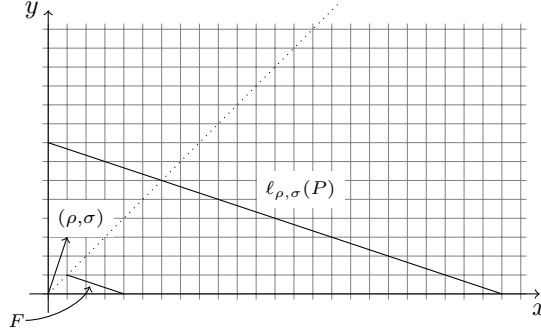
A SHORT AND ELEMENTARY PROOF OF JUNG'S THEOREM

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ABSTRACT. We give a short and elementary proof of Jung's theorem, which states that for a field K of characteristic zero the automorphisms of $K[x, y]$ are generated by elementary automorphisms and linear automorphisms.

Introduction

The theorem of Jung [5] states that if K is a field of characteristic zero, then any automorphism of $L := K[x, y]$ is the finite composition of elementary automorphisms (given by $x \mapsto x$, $y \mapsto y + p(x)$ or by $y \mapsto y$, $x \mapsto x + p(y)$) and linear automorphisms. Many authors have given proofs of this fact, for example in [1], [3], [7], [8], [9], [10] and [11], the last and very short and elegant proof in [6], which works in every algebraically closed field of characteristic zero. The key step in the proof of [6] is the same as in ours: In the situation of the figure,



there exists a polynomial F (called ζ in [6]) such that $F = \mu x(y + \lambda x^\sigma)$ and $[F, \ell_{\rho, \sigma}(P)] = \ell_{\rho, \sigma}(P)$. Then we apply φ given by $\varphi(x) := x$ and $\varphi(y) := y - \lambda x^\sigma$, and obtain $\deg(\varphi(P)) < \deg(P)$.

Here $[P, Q]$ stands for the determinant of the jacobian matrix of two polynomials P, Q and $\ell_{\rho, \sigma}(P)$ is the leading form of P with respect to the weight (ρ, σ) . To our knowledge our proof is the shortest and simplest, (except the proof of [6]), and Theorem 1.4 is the only fact that we use that is not straightforward or elementary. The element F can be traced back to 1975 in [4]. In order to obtain a proof for a field that is not necessarily algebraically closed, we have to prove that for a polynomial automorphism there can be only one point at infinity, which we do in Proposition 2.2.

2010 *Mathematics Subject Classification.* primary 13F25; secondary 13P15.

Key words and phrases. Jacobian Conjecture.

Supported by UBACYT 095, PIP 112-200801-00900 (CONICET) and PUCP-DGI-2013-3036.

Supported by UBACYT 095 and PIP 112-200801-00900 (CONICET).

Christian Valqui was supported by PUCP-DGI-2013-3036.

1 Preliminaries

We first gather notations and results of [2]. We define the set of directions by

$$\mathfrak{V} := \{(\rho, \sigma) \in \mathbb{Z}^2 : \gcd(\rho, \sigma) = 1\}.$$

For all $(\rho, \sigma) \in \mathfrak{V}$ and $(i, j) \in \mathbb{Z} \times \mathbb{Z}$ we write $v_{\rho, \sigma}(i, j) := \rho i + \sigma j$ and for $P = \sum a_{i, j} x^i y^j \in L \setminus \{0\}$, we define

- The *support* of P as $\text{Supp}(P) := \{(i, j) : a_{i, j} \neq 0\}$.
- The (ρ, σ) -*degree* of P as $v_{\rho, \sigma}(P) := \max \{v_{\rho, \sigma}(i, j) : a_{i, j} \neq 0\}$.
- The (ρ, σ) -*leading term* of P as $\ell_{\rho, \sigma}(P) := \sum_{\{\rho i + \sigma j = v_{\rho, \sigma}(P)\}} a_{i, j} x^i y^j$.

We say that $P \in L$ is (ρ, σ) -*homogeneous* if $P = \ell_{\rho, \sigma}(P)$. We assign to each direction its corresponding unit vector in S^1 , and we define an *interval* in \mathfrak{V} as the preimage under this map of an arc of S^1 that is not the whole circle. We consider each interval endowed with the order that increases counterclockwise.

For each $P \in L \setminus \{0\}$, we let $H(P)$ denote the *Newton polygon* of P , and it is evident that each one of its edges is the convex hull of the support of $\ell_{\rho, \sigma}(P)$, where (ρ, σ) is orthogonal to the given edge and points outside of $H(P)$. This directions form the set

$$\text{Dir}(P) := \{(\rho, \sigma) \in \mathfrak{V} : \#\text{Supp}(\ell_{\rho, \sigma}(P)) > 1\}.$$

Notation 1.1. Let $(\rho, \sigma) \in \mathfrak{V}$ arbitrary. We let $\text{st}_{\rho, \sigma}(P)$ and $\text{en}_{\rho, \sigma}(P)$ denote the first and the last point that we find on $H(\ell_{\rho, \sigma}(P))$ when we run counterclockwise along the boundary of $H(P)$. Note that these points coincide when $\ell_{\rho, \sigma}(P)$ is a monomial.

We say that two vectors $A, B \in \mathbb{R}^2$ are *aligned* and write $A \sim B$, if $0 = A \times B := \det \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$

Proposition 1.2 ([2, Proposition 2.4]). *Let $P, Q, R \in L \setminus \{0\}$ be such that*

$$[\ell_{\rho, \sigma}(P), \ell_{\rho, \sigma}(Q)] = \ell_{\rho, \sigma}(R),$$

where $(\rho, \sigma) \in \mathfrak{V}$. We have:

- (1) $\text{st}_{\rho, \sigma}(P) \approx \text{st}_{\rho, \sigma}(Q)$ *if and only if* $\text{st}_{\rho, \sigma}(P) + \text{st}_{\rho, \sigma}(Q) - (1, 1) = \text{st}_{\rho, \sigma}(R)$.
- (2) $\text{en}_{\rho, \sigma}(P) \approx \text{en}_{\rho, \sigma}(Q)$ *if and only if* $\text{en}_{\rho, \sigma}(P) + \text{en}_{\rho, \sigma}(Q) - (1, 1) = \text{en}_{\rho, \sigma}(R)$.

Remark 1.3. Let $(\rho, \sigma) \in \mathfrak{V}$ and let $P, F \in L$ be (ρ, σ) -homogeneous such that $[F, P] = P$. If F is a monomial, then $F = \lambda xy$ with $\lambda \in K^\times$, and, either $\rho + \sigma = 0$ or P is also a monomial.

The following theorem is an important tool in the constructions of [2]. It is the only result we use that is not straightforward.

Theorem 1.4 ([2, Theorem 2.6]). *Let $P \in L$ and let $(\rho, \sigma) \in \mathfrak{V}$ be such that $\rho + \sigma > 0$ and $v_{\rho, \sigma}(P) > 0$. If $[P, Q] \in K^\times$ for some $Q \in L$, then there exists a (ρ, σ) -homogeneous element $F \in L$ such that*

$$v_{\rho, \sigma}(F) = \rho + \sigma \quad \text{and} \quad [F, \ell_{\rho, \sigma}(P)] = \ell_{\rho, \sigma}(P). \quad (1.1)$$

If I is an interval in \mathfrak{V} and if there is no closed half circle contained in I , then for all $(\rho, \sigma), (\rho_1, \sigma_1) \in I$ we have

$$(\rho_1, \sigma_1) < (\rho, \sigma) \iff (\rho_1, \sigma_1) \times (\rho, \sigma) > 0. \quad (1.2)$$

Proposition 1.5 ([2, Proposition 3.7]). *Let $P \in L \setminus \{0\}$ and let (ρ_1, σ_1) and (ρ_2, σ_2) be consecutive elements in $\text{Dir}(P)$. If $(\rho_1, \sigma_1) < (\rho, \sigma) < (\rho_2, \sigma_2)$, then $\text{en}_{\rho_1, \sigma_1}(P) = \text{Supp}(\ell_{\rho, \sigma}(P)) = \text{st}_{\rho_2, \sigma_2}(P)$.*

Proposition 1.6 ([2, Proposition 3.10]). *Let $P, Q \in L$ and $\varphi: L \rightarrow L$ an algebra morphism. Then*

$$[\varphi(P), \varphi(Q)] = \varphi([P, Q])[\varphi(x), \varphi(y)]. \quad (1.3)$$

2 Jung's Theorem

Lemma 2.1. *Let $f: K[x, y] \rightarrow K[x, y]$ be an automorphism, set $P := f(x)$ and assume $(a, b) \in \{\text{en}_{1,1}(P), \text{st}_{1,1}(P)\}$. Then $a = 0$ or $b = 0$.*

Proof. We will proof only the case $(a, b) = \text{st}_{1,1}(P)$, since the argument in the other case is the same. Assume $a \geq b > 0$. We set $R_0 := x$ and $R_{j+1} := [R_j, P]$. Then $\text{st}_{1,1}(R_0) = (1, 0) \approx (a, b) = \text{st}_{1,1}(P)$ and so, by Proposition 1.2(1)

$$\text{st}_{1,1}(R_1) = (1, 0) + (a, b) - (1, 1) \approx (a, b).$$

Increasing k and using Proposition 1.2(1), one obtains inductively that

$$\text{st}_{1,1}(R_k) = (1, 0) + k(a, b) - k(1, 1) \approx (a, b),$$

since $((1, 0) + k(a, b) - k(1, 1)) \times (a, b) = b + k(a - b) \neq 0$. Consequently $R_k \neq 0$ for all k , which contradicts the fact that $R_n = 0$ for $n \gg 0$.

If $b \geq a > 0$, then we set $R_0 := y$ and $R_{j+1} := [R_j, P]$ and the same argument yields a contradiction. Hence $a = 0$ or $b = 0$, as desired. \square

The next proposition shows that for an automorphism f , there can be only one factor at infinity, or equivalently, that $\ell_{1,1}(f(x))$ is the power of one linear factor.

Proposition 2.2. *Let $f: K[x, y] \rightarrow K[x, y]$ be an automorphism and set $P := f(x)$. Then*

$$\text{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}, \quad \text{Supp}(\ell_{1,1}(P)) = \{(0, a)\} \quad \text{or} \quad \ell_{1,1}(P) = \mu(x - \lambda y)^a,$$

for some $\mu, \lambda \in K^\times$, where $a = v_{1,1}(P)$.

Proof. We can assume that K is algebraically closed. Let $a = v_{1,1}(P) > 0$ and write $\ell_{1,1}(P) = x^a p(z)$, where $z := x^{-1}y$ and $p(z) \in K[z]$. If $0 < b := \deg(p(z)) < a$, then $\text{en}_{1,1}(P) = (a, 0) + b(-1, 1) = (a - b, b)$ which contradicts Lemma 2.1. If $\deg(p) = 0$, then $\text{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}$, so it suffices to consider the case $\deg(p(z)) = a$. If neither

$$\text{Supp}(\ell_{1,1}(P)) = \{(0, a)\} \quad \text{nor} \quad \ell_{1,1}(P) = \mu(x - \lambda y)^a,$$

then $p(z) = \mu \prod_{i=1}^k (z - \lambda_i)^{m_i}$ has a root λ_{i_0} with multiplicity $0 < m_{i_0} < a$. But then the automorphism φ given by $\varphi(x) := x$ and $\varphi(y) := y + \lambda_{i_0}x$ yields

$$\ell_{1,1}(\varphi(P)) = \varphi(\ell_{1,1}(P)) = x^a p(z + \lambda_{i_0}) = \mu x^a z^{m_{i_0}} \prod_{\substack{i=1 \\ i \neq i_0}}^k (z - \bar{\lambda}_i)^{m_i},$$

where $\bar{\lambda}_i = \lambda_i - \lambda_{i_0}$ and $\prod_{\substack{i=1 \\ i \neq i_0}}^k \bar{\lambda}_i^{m_i} \neq 0$. This implies

$$\text{st}_{1,1}(\varphi(P)) = (a, 0) + m_{i_0}(-1, 1) = (a - m_{i_0}, m_{i_0}),$$

where $a - m_{i_0} \neq 0$ and $m_{i_0} \neq 0$, which contradicts Lemma 2.1 and concludes the proof. \square

Theorem 2.3. *Let $f : K[x, y] \rightarrow K[x, y]$ be an automorphism. Then f is the composition of elementary automorphisms and linear automorphisms.*

Proof. Set $P := f(x)$. If $\deg(P) = 1$, then we can assume $P = x$, and then $f(y) = \lambda y + q(x)$; it follows that f is the composition of elementary automorphisms and linear automorphisms.

Therefore it suffices to prove that either $\deg(P) = v_{1,1}(P) = 1$ or there exists an elementary automorphism φ such that $\deg(\varphi(P)) < \deg(P)$. By Proposition 2.2 we can assume

$$\text{Supp}(\ell_{1,1}(P)) = \{(a, 0)\}$$

for some $a \in \mathbb{N}$. In fact, if $\text{Supp}(\ell_{1,1}(P) = \{(0, a)\}$ we apply the flip, which is a composition of elementary automorphisms, and if $\ell_{1,1}(P) = \mu(x - \lambda y)^a$, then we apply the elementary automorphism given by $x \mapsto x + \lambda y$ and $y \mapsto y$.

We also can assume that P is not a monomial and so we set $(\rho, \sigma) := \text{Succ}_P(1, 1)$, the *successor* of $(1, 1)$, which is the first element of $\text{Dir}(P)$ that one encounters starting from $(1, 1)$ and running counterclockwise.

If $(\rho, \sigma) \geq (0, 1)$, then from Proposition 1.5 we obtain $(a, 0) = \text{st}_{0,1}(P)$ and then for all $(i, j) \in \text{Supp}(P)$ we have $j = v_{0,1}(i, j) \leq v_{0,1}(a, 0) = 0$, hence $P \in K[x]$. It follows easily that $1 = \deg(P)$.

It remains to consider the case $(1, 1) < (\rho, \sigma) < (0, 1)$, which implies $\sigma > \rho > 0$. By Theorem 1.4 there exist a (ρ, σ) -homogenous element $F \in K[x, y]$ such that

$$[F, \ell_{\rho,\sigma}(P)] = \ell_{\rho,\sigma}(P) \quad \text{and} \quad v_{\rho,\sigma}(F) = \rho + \sigma.$$

For all $(i, j) \in \text{Supp}(F)$ we have $\rho i + \sigma j = \rho + \sigma$ and so

$$(1 - i)\rho = (j - 1)\sigma.$$

Hence $j > 1$ is impossible and if $j = 1$, then $i = 1$. Since $\ell_{\rho,\sigma}(P)$ is not a monomial, we know from Remark 1.3 that F has at least two points in its support, hence $(1, 1) \in \text{Supp}(F)$ and there must be a point of the form $(i, 0) \in \text{Supp}(F)$. But then $\sigma = (i - 1)\rho$ and we obtain $\rho = 1$, since ρ and σ are coprime. We obtain that $F = \mu x(y + \lambda x^\sigma)$ for some $\mu, \lambda \in K^\times$ and $\ell_{\rho,\sigma}(P) = x^a p(z)$ for some $p(z) \in K[z]$ with $N := \deg(p(z)) > 0$, where $z := yx^{-\sigma}$.

Consider now the elementary automorphism φ given by $\varphi(x) := x$ and $\varphi(y) := y - \lambda x^\sigma$. Since φ is (ρ, σ) -homogenous we have $\varphi(\ell_{\rho,\sigma}(P)) = \ell_{\rho,\sigma}(\varphi(P))$ and so, by Proposition 1.6, we obtain

$$[\varphi(F), \ell_{\rho,\sigma}(\varphi(P))] = [\varphi(F), \varphi(\ell_{\rho,\sigma}(P))] = \varphi(\ell_{\rho,\sigma}(P)) = \ell_{\rho,\sigma}(\varphi(P)).$$

Moreover $\varphi(F) = \mu xy$ is a monomial, hence $\ell_{\rho,\sigma}(\varphi(P))$ is also a monomial. It follows that $\ell_{\rho,\sigma}(\varphi(P)) = \mu_p x^a z^N$, with $N = \deg(p(z)) > 0$, since $\ell_{\rho,\sigma}(\varphi(P)) = \varphi(x^a p(z)) = x^a p(z - \lambda)$ (Note that $\varphi(z) = z - \lambda$). Therefore we arrive at $(a, 0) \notin \text{Supp}(\varphi(P))$.

Now, for $(i, j) \in \text{Supp}(\varphi(P))$ we have

$$v_{1,1}(i, j) = i + j \leq i + \sigma j = v_{\rho,\sigma}(i, j) \leq v_{\rho,\sigma}(\varphi(P)) = v_{\rho,\sigma}(P) = v_{\rho,\sigma}(a, 0) = a = v_{1,1}(P),$$

and the equality would be possible only if $j = 0$ and $i = a$, but we have $(i, j) \neq (a, 0)$. Hence $v_{1,1}(\varphi(P)) < v_{1,1}(P)$, as desired. \square

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